

SELECTIVE GAME VERSIONS OF COUNTABLE TIGHTNESS WITH BOUNDED FINITE SELECTIONS

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ABSTRACT. For a topological space X and a point $x \in X$, consider the following game – related to the property of X being countably tight at x . In each inning $n \in \omega$, the first player chooses a set A_n that clusters at x , and then the second player picks a point $a_n \in A_n$; the second player is the winner if and only if $x \in \overline{\{a_n : n \in \omega\}}$.

In this work, we study variations of this game in which the second player is allowed to choose finitely many points per inning rather than one, but in which the number of points they are allowed to choose in each inning has been fixed in advance. Surprisingly, if the number of points allowed per inning is the same throughout the play, then all of the games obtained in this fashion are distinct. We also show that a new game is obtained if the number of points the second player is allowed to pick increases at each inning.

1. INTRODUCTION

Countable tightness is a classical topological property introduced in [18] as a generalization of first-countability: a topological space X is *countably tight* at a point $x \in X$ if every $A \subseteq X$ with $x \in \overline{A}$ has a countable subset C with $x \in \overline{C}$. In other words, X is countably tight at x if the countable sets in $\Omega_p = \{A \subseteq X : x \in \overline{A}\}$ constitute a cofinal family in Ω_p with respect to the order \supseteq . Loosely speaking, in a countably tight space one can study clustering properties – hence the topology itself – just by looking at the countable subsets of the space.

It is well-known that the product of two countably tight topological spaces need not be countably tight (see e.g. [1]), although this will be the case if one of them is a first-countable space. Thus, when dealing with countable tightness in topological products, it is natural to consider the class of productively countably tight spaces: a topological space X is *productively countably tight* at $x \in X$ if, for every topological space Y that is countably tight at a point $y \in Y$, the product space $X \times Y$ is countably tight at the point (x, y) . In [2], A. V. Arkhangel'skiĭ obtained an internal characterization for the productively countably tight Tychonoff spaces (Theorem 2.7 below). In [12] and, more recently, in [4], connections between productivity of countable tightness and selective topological properties and their game versions have arisen (see Theorem 1.1); these will be the starting point of our investigations in this work.

A topological space X has *countable strong fan tightness* at a point $x \in X$ [22] if, for every sequence $(A_n)_{n \in \omega}$ of elements of Ω_x , we can select $a_n \in A_n$ for $n \in \omega$

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in such a way that $\{a_n : n \in \omega\} \in \Omega_x$. Countable strong fan tightness may be regarded as a selective version of countable tightness, in a combinatorial sense; rather than just stating that every $A \in \Omega_x$ includes a countable subset also in Ω_x , we now require that a new element of Ω_x can be obtained by putting together points selected from countably many elements of Ω_x . For reasons that will become clearer later, we will adopt Scheepers's notation for selective topological properties [24] and refer to countable strong fan tightness at x as $S_1(\Omega_x, \Omega_x)$.

The property $S_1(\Omega_x, \Omega_x)$ has a combinatorial game of infinite length naturally associated to it [25], which we denote by $G_1(\Omega_x, \Omega_x)$. This game is played between the players ONE and TWO according to the following rules. In each inning $n \in \omega$, ONE chooses a member $A_n \in \Omega_x$, and then TWO picks a point $a_n \in A_n$. The winner is TWO if $\{a_n : n \in \omega\} \in \Omega_x$, and ONE otherwise. If P is a player of this game, we will write $P \uparrow G_1(\Omega_x, \Omega_x)$ instead of “ P has a winning strategy in $G_1(\Omega_x, \Omega_x)$ ” – and this notational convention will be extended to all of the other games we will consider in this work.

It is clear that, for every space X and every $x \in X$,

$$\text{TWO} \uparrow G_1(\Omega_x, \Omega_x) \implies \text{ONE} \nmid G_1(\Omega_x, \Omega_x) \implies S_1(\Omega_x, \Omega_x).$$

As it turns out – by putting together Theorems 3.9 and 4.2 of [12]¹ and Theorem 2.7 of [4] –, the property of being productively countably tight also lies between $\text{TWO} \uparrow G_1(\Omega_x, \Omega_x)$ and $S_1(\Omega_x, \Omega_x)$, at least for Tychonoff spaces:

Theorem 1.1 (Gruenhage [12], Aurichi–Bella [4]). *Let X be a topological space and $x \in X$.*

- (a) *If $\text{TWO} \uparrow G_1(\Omega_x, \Omega_x)$ on X , then X is productively countably tight at x .*
- (b) *If X is a Tychonoff space and is productively countably tight at x , then $S_1(\Omega_x, \Omega_x)$ holds in X .*

The issues we address in Section 2 refer to the properties mentioned above. Answering Question 2.9 of [4], we show (in Example 2.4) that productivity of countable tightness does not imply $\text{ONE} \nmid G_1(\Omega_x, \Omega_x)$. The (rather unexpected) fact that the counterexample considered has a different behaviour with respect to a seemingly minor variation of the game $G_1(\Omega_x, \Omega_x)$ prompts us to investigate variations of this game in which TWO may pick more than one point per inning. This will be our main interest in Section 3, in which we show that, surprisingly, each finite bound in the number of points TWO is allowed to pick per inning leads to a different game. We conclude in Section 4 with a discussion on possible new directions that can be pursued from the results obtained in this work.

A word on notation. The set of the natural numbers is denoted by ω , and we write \mathbb{N} in place of $\omega \setminus \{0\}$. For a set A , the symbol $^{<\omega}A$ stands for the set of all of the finite sequences of elements of A ; furthermore, we write $^{\leq\omega}A$ instead of $(^{<\omega}A) \cup (^\omega A)$. Finally,

- $[A]^n$ denotes the set of all of the subsets of A of cardinality n , for a given $n \in \omega$;
- $[A]^{<\aleph_0}$ denotes $\bigcup_{n \in \omega} [A]^n$;
- $[A]^{\aleph_0}$ denotes the set of all of the countable infinite subsets of A ;
- $[A]^{\leq\aleph_0}$ denotes $[A]^{<\aleph_0} \cup [A]^{\aleph_0}$.

¹Although the game considered in [12] appears to be quite different from $G_1(\Omega_x, \Omega_x)$, these two games are closely related – see Remark 3.11.

All of the topological spaces we consider in this text are assumed to be T_1 .

2. SOME EXAMPLES CONCERNING $S_1(\Omega_x, \Omega_x)$, $G_1(\Omega_x, \Omega_x)$ AND PRODUCTIVITY OF COUNTABLE TIGHTNESS

We begin this section with a couple of results witnessing some differences between the properties considered in the Introduction. From Theorem 1.1, we know that, if $\text{Two} \uparrow G_1(\Omega_x, \Omega_x)$ on some topological space X , then X is productively countably tight at x . An important class of spaces show that this implication cannot be reversed: the one-point compactifications of Mrówka spaces [19].

Proposition 2.1. *Let \mathcal{A} be an uncountable almost disjoint family on ω and $\Psi(\mathcal{A})$ be the corresponding Mrówka space. Let $X = \{p\} \dot{\cup} \Psi(\mathcal{A})$ be the one-point compactification of $\Psi(\mathcal{A})$. Then X is a compact space that is productively countably tight at the point p , yet $\text{Two} \nmid G_1(\Omega_p, \Omega_p)$ on X .*

Proof. Since X is a compact space of countable tightness, it follows from Theorem 4 of [16] that X is productively countably tight at p . Since X is separable but not first-countable, Theorem 3.6 of [12] tells us that Two does not have a winning strategy in $G_1(\Omega_p, \Omega_p)$. \square

We will see later on (in Example 2.10) that for a specific case of the previous proposition we can even obtain $\text{One} \uparrow G_1(\Omega_p, \Omega_p)$.

In view of the two chains of implications mentioned in the Introduction, it is natural to ask the relation between being productively countably tightness at a point x and $S_1(\Omega_x, \Omega_x)$ for Tychonoff spaces. A class of spaces shows that one of the implications is not true: the spaces of continuous real-valued functions defined on a first-countable uncountable γ space [29, Theorem 6].

Proposition 2.2. *Let $X = C_p(Y)$, where Y is a first-countable uncountable γ space. Then $\text{One} \nmid G_1(\Omega_x, \Omega_x)$ on X , yet X is not productively countably tight at x .*

Proof. It follows from results of Gerlits and Nagy [11, Theorem 2] and Sharma [26, Theorem 1] that $\text{One} \nmid G_1(\Omega_f, \Omega_f)$ for each $f \in C_p(Y)$; however, $C_p(Y)$ is not productively countably tight since the G_δ -modification of Y is not Lindelöf [31, Theorem 1]. \square

The other implication was a question in a paper of the first- and the second-named authors:

Question 2.3 (Aurichi–Bella [4]). *Let X be productively countably tight at $x \in X$. Is it true that $\text{One} \nmid G_1(\Omega_x, \Omega_x)$ on X ?*

In order to obtain a negative answer to Question 2.3 in a stronger sense, we evoke an example from [25].

Here we consider the game $G_{\text{fin}}(\Omega_x, \Omega_x)$, which is a standard variation of $G_1(\Omega_x, \Omega_x)$ – see [25]. In each inning $n \in \omega$ of $G_{\text{fin}}(\Omega_x, \Omega_x)$, One chooses $A_n \in \Omega_x$, and then Two chooses a finite subset $F_n \subseteq A_n$. The winner is Two if $\bigcup_{n \in \omega} F_n \in \Omega_x$, and One otherwise.

Example 2.4. *There exists a countable space X with only one non-isolated point p that is productively countably tight at p , but on which $\text{One} \uparrow G_{\text{fin}}(\Omega_p, \Omega_p)$.*

Proof. Let $X = \omega \dot{\cup} \{p\}$ be the space in [25, pp. 250–251], defined as follows. We first construct the function F that will be a winning strategy for ONE in $G_{\text{fin}}(\Omega_p, \Omega_p)$, and then we make use of F to define the topology of X .

Fix a partition $\{Y_n : n \in \omega\}$ of ω in infinite sets. Now construct a strategy F for ONE such that:

- (1) for every sequence of non-empty finite sets $N_1, \dots, N_k \subseteq \omega$ that are subsets of different Y_j s, there is an i for which $F(N_1, \dots, N_k) = Y_i$;
- (2) for every i , there is a unique sequence of finite sets N_1, \dots, N_k such that:
 - (a) each N_j is a non-empty finite subset of some Y_{k_j} with $k_j \neq i$;
 - (b) if $j \neq m$, then $k_j \neq k_m$;
 - (c) $F(N_1, \dots, N_k) = Y_i$.

A way to define F can be the following. Let $\{T_n : n \in \omega\}$ be an injective enumeration of all finite sequences of non-empty finite sets lying in different Y_j s, and let I_n be the set of those $j \in \omega$ such that Y_j contains an element in the sequence T_n . By induction, for each $n \in \omega$ let $i = \min(\omega \setminus (K \cup I_n))$, where K is the set of those $j \in \omega$ such that $F(T_m) = Y_j$ for $m < n$. Then, put $F(T_n) = Y_i$.

For each play $P = (O_1, N_1, O_2, N_2, \dots)$ of $G_{\text{fin}}(\Omega_p, \Omega_p)$ in X , we put $S(P) = \bigcup_{k \in \omega} N_k$.

Every point of ω is isolated in X , while a local base at p in X consists of sets of the form $V(G) = X \setminus \bigcup_{P \in G} S(P)$ for G a finite set of plays in which ONE uses the strategy F .

Fact 2.5 ([25]). $\text{ONE} \uparrow G_{\text{fin}}(\Omega_p, \Omega_p)$ on X .

Fact 2.6. X is productively countably tight at p .

In order to prove Fact 2.6, we will make use of a result obtained in Theorem 3.5 of [2]. Recall that a family \mathcal{F} of subsets of a space Y is a π -network at a point $y \in Y$ if every open neighbourhood of y includes some element of \mathcal{F} , and that a family of sets is *centred* if each of its nonempty finite subfamilies has nonempty intersection.

Theorem 2.7 (Arkhangel'skiĭ [2]). *Let Y be a Tychonoff space and $y \in Y$. The following assertions are equivalent:*

- (a) Y is productively countably tight at y ;
- (b) whenever $\bigcup_{i \in I} \mathcal{C}_i \subseteq [Y]^{\leq \aleph_0}$ is a π -network at y such that each \mathcal{C}_i is nonempty and centred, there is a countable $I^* \subseteq I$ such that $y \in \overline{\bigcup_{i \in I^*} A_i}$ for every choice of $A_i \in \mathcal{C}_i$, $i \in I^*$.

We will show that condition (b) holds for X at p . Let $\bigcup_{i \in I} \mathcal{C}_i \subseteq [X]^{\leq \aleph_0}$ be a π -network at p such that each \mathcal{C}_i is nonempty and centred. Since X is a regular space, we can assume that each \mathcal{C}_i consists of closed sets. We may further assume that $p \notin \bigcap \mathcal{C}_i$ for every $i \in I$, since an $i' \in I$ satisfying $\forall A \in \mathcal{C}_{i'} (p \in A)$ would immediately imply the required condition with $I^* = \{i'\}$.

Lemma 2.8. *Let P be a play during which player ONE uses F . If S is infinite and $S \subseteq S(P)$, then P is uniquely determined.*

Proof. The key point is that, if P and P' are distinct plays in which ONE uses F , then $S(P) \cap S(P')$ must be finite. Let $P = (O_1, N_1, O_2, N_2, \dots)$ and $P' = (O'_1, N'_1, O'_2, N'_2, \dots)$, and let k be the least integer for which $N_k \neq N'_k$. Taking into account property (2) of F , we have $F(N_1, \dots, N_k) \neq F(N'_1, \dots, N'_k)$

and consequently $N_{k+1} \cap N'_{k+1} = \emptyset$. Next, we see that the sets $F(N_1, \dots, N_k)$, $F(N_1, \dots, N_{k+1})$, $F(N'_1, \dots, N'_k)$, $F(N'_1, \dots, N'_{k+1})$ are mutually distinct, and consequently the sets $N_{k+1}, N_{k+2}, N'_{k+1}, N'_{k+2}$ are pairwise disjoint. By continuing to argue in this manner, we conclude that $S(P) \cap S(P') = N_1 \cup \dots \cup N_{k-1}$. \square

Lemma 2.9. *For any open set $U \subseteq X$ with $p \in U$ there exists a countable set $I' \subseteq I$ and pairwise disjoint elements $A_i \in \mathcal{C}_i$ satisfying $A_i \subseteq U$ for $i \in I'$.*

Proof. This follows easily from $\bigcup_{i \in I} \mathcal{C}_i$ being a π -network of X consisting of closed sets, together with the fact that $p \notin \bigcap \mathcal{C}_i$ for each $i \in I$. \square

Proof of Fact 2.6. Using Lemma 2.9, take a countable set $I'_0 \subset I$ and pairwise disjoint elements $A_i \in \mathcal{C}_i$ for $i \in I'_0$. If for any $B_i \in \mathcal{C}_i$, $i \in I'_0$, we have $p \in \overline{\bigcup_{i \in I'_0} B_i}$, then we stop. Otherwise, since the families \mathcal{C}_i are centred, we may assume that $p \notin \overline{\bigcup_{i \in I'_0} A_i}$. Then we fix a finite set G_0 of plays in which ONE uses the strategy F such that $\bigcup_{i \in I'_0} A_i \cap V(G_0) = \emptyset$. Applying again Lemma 2.9 to the open set $V(G_0)$, we find a countable $I'_1 \subseteq I$ and a pairwise disjoint family $\{A_i \in \mathcal{C}_i : i \in I'_1\}$ such that $A_i \subseteq V(G_0)$ for each $i \in I'_1$. If I'_1 is good for our purpose, then we stop. Otherwise, we fix a finite set G_1 of plays in which ONE uses F such that $\bigcup_{i \in I'_1} A_i \cap V(G_1) = \emptyset$. Of course, we may choose the set G_1 disjoint from G_0 . Then, we continue by working in the open set $V(G_0 \cup G_1)$. If the process never stops, at the end we put $I^* = \bigcup_{n \in \omega} I'_n$. We claim that the countable set I^* satisfies condition (b) of Theorem 2.7. To this end, take $B_i \in \mathcal{C}_i$ for each $i \in I^*$ and choose a point $s_i \in A_i \cap B_i$ for each $i \in I^*$. It suffices to check that $p \in \overline{\{s_i : i \in I^*\}}$. By contradiction, assume that there is a finite set H of plays in which ONE makes use of F such that $\{s_i : i \in I^*\} \subseteq \bigcup_{P \in H} S(P)$. Since the set $S_n = \{s_i : i \in I'_n\}$ is infinite, there exists some $P_n \in H$ such that the set $S_n \cap S(P_n)$ is infinite. Again by Lemma 2.8, we must have $P_n \in G_n$. But, according to our construction, the sets in $\{G_n : n \in \omega\}$ are pairwise disjoint and consequently the plays P_n should be mutually distinct. This is obviously impossible and we are done. \square

Fact 2.6 finishes the proof of Example 2.4. \square

One of the nicest classes of productively countably tight spaces is the class of the compact spaces of countable tightness [16, Theorem 4]. One could expect to have a positive answer to Question 2.3 in this class. But, again, this is not the case.

Example 2.10. *There exists a compact Hausdorff space of countable tightness X and a point $p \in X$ such that $\text{ONE} \uparrow G_1(\Omega_p, \Omega_p)$.*

Proof. This is essentially a particular case of Proposition 2.1. Endow the set ${}^{\leq \omega} \omega$ with the topology in which every point of ${}^{< \omega} \omega$ is isolated and basic neighbourhoods of $f \in {}^\omega \omega$ are of the form $\{f\} \cup \{f \upharpoonright j : j \in \omega \setminus k\}$ for $k \in \omega$. Let $X = \{p\} \dot{\cup} {}^{\leq \omega} \omega$ be the one-point compactification of this space. Let ONE's first move in $G_1(\Omega_p, \Omega_p)$ be $\{(k) : k \in \omega\}$ and, in general, if TWO picks a point $s \in {}^{< \omega} \omega$, let ONE's move in the next inning be $\{s^\frown(k) : k \in \omega\} \in \Omega_p$. It is clear that this is a winning strategy for ONE in $G_1(\Omega_p, \Omega_p)$, since all of TWO's moves lie on a single branch of the tree ${}^{< \omega} \omega$. \square

After the last two examples, an obvious further question emerges:

Problem 2.11. *Can a compact space of countable tightness X have a point x such that $\text{ONE} \uparrow G_{\text{fin}}(\Omega_x, \Omega_x)$?*

Notice that Example 2.10 cannot be used to get a positive answer to Problem 2.11. Indeed, the next observation shows that even in compact spaces the games $G_1(\Omega_p, \Omega_p)$ and $G_{\text{fin}}(\Omega_p, \Omega_p)$ can have a very distinct behaviour (we will see further differences between games of this kind in the next section).

Proposition 2.12. *If X is the compact space in Example 2.10, then $\text{Two} \uparrow G_2(\Omega_p, \Omega_p)$.*

Proof. We will prove that TWO can reply to ONE in such a way that, for each n , the set of all the answers played by TWO in the first n innings includes a set $\{s_1, \dots, s_n\}$ with the property that no branch contains two elements of it. Note that, in this way, TWO wins the game.

We will do so by induction. With no loss of generality, we can assume that ONE does not play a set containing points of ω^ω . If, in the first inning, ONE plays A_1 , then TWO chooses $\{s_1, s_2\} \subset A_1$ such that s_1 and s_2 are not in the same branch. Suppose that at the end of the n -th inning, the set of all answers of TWO contains a set $\{s_1, \dots, s_n\}$ with the prescribed property. Let A_{n+1} be the move of ONE in the inning $n+1$. If there is a point in A_{n+1} that lies in a branch missing $\{s_1, \dots, s_n\}$, then TWO chooses this point together with some other one. In the remaining case, since p is in the closure of A_{n+1} , there is at least one s_i and two incompatible elements $a_1, a_2 \in A_{n+1}$ such that $s_i \subset a_1$ and $s_i \subset a_2$. The answer of TWO in the $(n+1)$ -th inning will be just $\{a_1, a_2\}$. Observe that every branch meets the set $\{s_j : j \neq i\} \cup \{a_1, a_2\}$ in at most one point. \square

In view of Theorem 1.1, one may wonder what the real strength of the property $\text{Two} \uparrow G_2(\Omega_x, \Omega_x)$ is.

Problem 2.13. *Let X be a space and $x \in X$. Does $\text{Two} \uparrow G_2(\Omega_x, \Omega_x)$ imply that X is productively countably tight at x ?*

A simple example of a countable space that is not productively countably tight is $\omega \cup \{p\} \subseteq \beta\omega$, for an arbitrary $p \in \beta\omega \setminus \omega$ [4]. Still in [4], it was further observed that, if p is a selective ultrafilter, then $S_1(\Omega_p, \Omega_p)$ holds in $\omega \cup \{p\}$. The possibility that such a space could provide a negative answer to Problem 2.13 is ruled out by the next fact:

Proposition 2.14 ([7], Proposition 3). *Let X be a space and x be a non-isolated point of X . If $\text{Two} \uparrow G_{\text{fin}}(\Omega_x, \Omega_x)$, then x is in the closure of two disjoint subsets of $X \setminus \{x\}$.*

Here we will show a little more: there are actually countably many disjoint sets that have x in their closures. This fact will follow from the next more general result.

Proposition 2.15. *Let X be a space on which $\text{Two} \uparrow G_{\text{fin}}(\Omega_x, \Omega_x)$ for a non-isolated point x . Then there is an almost disjoint family of cardinality 2^{\aleph_0} of elements of Ω_x .*

Proof. Let $\{s_n : n \in \omega\}$ be a one-to-one enumeration of $2^{<\omega}$ such that, if $s_m \subseteq s_n$, then $m \leq n$. Fix a winning strategy φ for TWO in $G_{\text{fin}}(\Omega_x, \Omega_x)$. Let $Q_{s_0} = X \setminus \{x\}$ and $A_{s_0} = \varphi(Q_{s_0})$. For each $n \in \omega$, define $Q_{s_n} = X \setminus (A_{s_0} \cup \dots \cup A_{s_{n-1}} \cup \{x\})$ and $A_{s_n} = \varphi(Q_{s_n} \restriction 0, \dots, Q_{s_n} \restriction (\text{dom}(s_n)-1), Q_{s_n})$. Since φ is a winning strategy, the set $B_g = \bigcup_{k \in \omega} A_{g \restriction k}$ is an element of Ω_x for each $g \in 2^\omega$. Note that, if $g, h \in 2^\omega$ are distinct, then $B_g \cap B_h = \bigcup_{j \leq k_0} A_{g \restriction j} \in [X]^{<\aleph_0}$, where $k_0 = \min\{k \in \omega : g(k) \neq h(k)\}$. \square

Recall that a space Y is *strongly Fréchet* [27] (or *countably bisquential* [17]) at a point y if for any decreasing sequence $(A_n)_{n \in \omega}$ of elements of Ω_y we may pick points $a_n \in A_n$ in such a way that $(a_n)_{n \in \omega}$ is a sequence converging to y .

In general, a space productively countably tight at a point p need not be Fréchet at p , as Proposition 2.1 above illustrates.

Fact 2.16. *The space X in Example 2.10 is strongly Fréchet at p .*

Proof. It follows from (16)(b) in [28] (see also [3, Proposition 3]) that a regular space Y is strongly Fréchet at a point y if and only if Y is Fréchet at y and $S_1(\Omega_y, \Omega_y)$ holds in Y . Thus, as the space X is productively countably tight and hence satisfies $S_1(\Omega_p, \Omega_p)$ [4], it suffices to check that X is Fréchet at p .

Let then $A \subseteq {}^{\leq \omega} \omega$ be such that $p \in \overline{A}$. If $|A \cap {}^\omega \omega| \geq \aleph_0$, then any injective function from ω into $A \cap {}^\omega \omega$ is a sequence that converges to p . We may then assume that $A \subseteq {}^{<\omega} \omega$. Thus, in order to conclude that there is a sequence of points of A converging to p , it suffices to show that A includes an infinite antichain, since any injective function from ω onto an antichain of X is a sequence that converges to p .

Let $Y = \{s \in A : \exists u, v \in A (s \subseteq u \ \& \ s \subseteq v \ \& \ u \perp v)\}$. We will consider two cases:

Case 1. Y is finite.

It follows from the definition of the set Y that, for each $t \in A \setminus Y$, there is $f_t \in {}^\omega \omega$ such that every $r \in A$ with $t \subseteq r$ satisfies $r \subseteq f_t$. Note that the set $A \setminus Y$ is the union of disjoint maximal chains. Since Y is finite and $p \in \overline{A}$, these maximal chains are infinitely many. Therefore, by picking a point in each of these maximal chains, we get an infinite antichain.

Case 2. Y is infinite.

If every chain included in Y is finite, then $\{\max_{\subseteq}(C) : C \text{ is a maximal chain included in } Y\}$ is an infinite antichain. Let us then assume that Y includes an infinite chain. Thus, let $f \in {}^\omega \omega$ and $D \in [\omega]^{\aleph_0}$ be such that $\{f \upharpoonright j : j \in D\} \subseteq Y$. First let $j_0 = \min(D)$ and $s_0 = f \upharpoonright j_0$. As $s_0 \in Y$, we can pick $t_0 \in A$ such that $s_0 \subsetneq t_0$ and $t_0 \not\subseteq f$. Now let $j_1 = \min(D \setminus \text{dom}(t_0))$ and $s_1 = f \upharpoonright j_1$, and pick $t_1 \in A$ with $s_1 \subsetneq t_1$ and $t_1 \not\subseteq f$. Let then $j_2 = \min(D \setminus \text{dom}(t_1))$ and $s_2 = f \upharpoonright j_2$, and so forth. By proceeding in this fashion, we construct an infinite antichain $\{t_n : n \in \omega\} \subseteq A$. \square

It is easy to check that a space Y is strongly Fréchet at y if and only if for every family $\{A_n : n \in \omega\} \subseteq \Omega_y$ we may pick points $a_n \in A_n$ in such a way that the set $\{a_n : n \in \omega\}$ contains a subsequence converging to y . To require that the entire set $\{a_n : n \in \omega\}$ is a sequence converging to y is a much stronger condition, which is called *strictly Fréchet at y* in [11].

Another interesting feature of the space X in Example 2.10 is that it is strongly Fréchet at p , but not strictly Fréchet at p . This is an immediate consequence of the following corollary to Theorem 1 of [26] and Theorem 3.9 of [12] – see also Remark 3.11 in the next section.

Theorem 2.17 (Galvin–Gruenhage–Sharma). *Let Y be a space and $y \in Y$. If Y is strictly Fréchet at y , then $\text{ONE} \nVdash \mathbf{G}_1(\Omega_y, \Omega_y)$.*

It is worth remarking that the arguments concerning strong and strict Fréchetness of the space in Example 2.10 apply equally to the space in Example 2.4.

Although Question 2.3 has a negative answer even for compact spaces of countable tightness, there is yet another relevant class of spaces to consider.

Problem 2.18. *Let X be a space that is bisequential at a point x . Is it true that $\text{ONE} \nVdash \mathbf{G}_1(\Omega_x, \Omega_x)$?*

Recall that a space Y is *bisequential* at a point y [17] if, for every filter base \mathcal{F} on Y that accumulates in y , there is a countable filter base \mathcal{G} that converges to y such that $F \cap G \neq \emptyset$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$. In [2], it was shown that, if Y is bisequential at y , then Y is productively countably tight at y .

As an attempt to answer Problem 2.18 in the negative, we may ask:

Problem 2.19. *Is the space X from Example 2.10 bisequential at p ?*

3. THE DIFFERENCES BETWEEN VARIOUS GAMES

In this section, we concentrate on investigating the differences that show up in several selective games related to countable tightness according to the number of points that player TWO is allowed to select in each inning.

We have already seen (in Proposition 2.12) that we obtain a different game if TWO is allowed to pick two points per inning instead of one. Now we will extend this result not only to an arbitrary fixed quantity of points, but also for a quantity that varies on the number of the inning being played.

We first recall the following result from [10, Section 3]. Here, for a function $f \in {}^\omega\mathbb{N}$, the notation $\mathbf{S}_f(\Omega_x, \Omega_x)$ stands for the following property: for every sequence $(A_n)_{n \in \omega}$ of elements of Ω_x , we can pick subsets $F_n \subseteq A_n$ for $n \in \omega$ in such a way that $|F_n| \leq f(n)$ for each $n \in \omega$ and $\bigcup_{n \in \omega} F_n \in \Omega_x$. This is a natural generalization of $\mathbf{S}_1(\Omega_x, \Omega_x)$, hence a (formally) more general selective version of countable tightness in combinatorial sense: rather than picking one point out of each A_n , we can now select up to $f(n)$ many points and, joining all of those points together (for all $n \in \omega$), we must assemble a new element of the family Ω_x . The following result, together with Example 3.5, basically shows that, for an unbounded function $f \in {}^\omega\mathbb{N}$, $\mathbf{S}_f(\Omega_x, \Omega_x)$ is indeed a new property — but only one new property is given in this fashion, since it does not matter which unbounded function is taken.

Proposition 3.1 (García-Ferreira–Tamariz-Mascarúa [10]). *Let X be a space and $x \in X$.*

- (a) *If $f \in {}^\omega\mathbb{N}$ is bounded, then $\mathbf{S}_f(\Omega_x, \Omega_x)$ is equivalent to $\mathbf{S}_1(\Omega_x, \Omega_x)$.*
- (b) *If $f, g \in {}^\omega\mathbb{N}$ are both unbounded, then $\mathbf{S}_f(\Omega_x, \Omega_x)$ is equivalent to $\mathbf{S}_g(\Omega_x, \Omega_x)$.*

We can now consider, for a given $f \in {}^\omega\mathbb{N}$, the game $\mathbf{G}_f(\Omega_x, \Omega_x)$ naturally associated to the selective property $\mathbf{S}_f(\Omega_x, \Omega_x)$. In each inning $n \in \omega$ of this game, ONE chooses $A_n \in \Omega_x$, and then TWO selects $F_n \subseteq A_n$ with $|F_n| \leq f(n)$. The winner is TWO if $\bigcup_{n \in \omega} F_n \in \Omega_x$, and ONE otherwise.

For $k \in \mathbb{N}$, we will write $\mathbf{G}_k(\Omega_x, \Omega_x)$ instead of $\mathbf{G}_{\bar{f}_k}(\Omega_x, \Omega_x)$, where $\bar{f}_k \in {}^\omega\mathbb{N}$ is the constant function having range $\{k\}$.

Corollary 3.2. *Let X be a space and $x \in X$. If $\text{TWO} \uparrow \mathbf{G}_k(\Omega_x, \Omega_x)$ on X for some $k \geq 1$, then $\mathbf{S}_1(\Omega_x, \Omega_x)$ holds.*

The above corollary shows that, in general, $\text{TWO} \uparrow \mathbf{G}_{\text{fin}}(\Omega_x, \Omega_x)$ is strictly weaker than $\text{TWO} \uparrow \mathbf{G}_k(\Omega_x, \Omega_x)$. To see this, it suffices to observe that $\text{TWO} \uparrow \mathbf{G}_{\text{fin}}(\Omega_0, \Omega_0)$ on $C_p(\mathbb{R})$ (see [5, Theorem 3.6] for a direct proof), but $\mathbf{S}_1(\Omega_0, \Omega_0)$ fails for $C_p(\mathbb{R})$ [22, Theorem 1]. $C_p(\mathbb{R})$ also shows that $\text{TWO} \uparrow \mathbf{G}_{\text{fin}}(\Omega_x, \Omega_x)$ does not imply productive countable tightness at x (see [31, Theorem 1]).

The next result, which is a game version of Proposition 3.1, is a first step towards drawing another line between the games $G_k(\Omega_x, \Omega_x)$ and $G_{\text{fin}}(\Omega_x, \Omega_x)$.

Proposition 3.3. *Let X be a space and $x \in X$.*

- (a) *If $f \in {}^\omega\mathbb{N}$ is bounded, then the games $G_f(\Omega_x, \Omega_x)$ and $G_k(\Omega_x, \Omega_x)$ are equivalent, where $k = \limsup_{n \in \omega} f(n) \in \mathbb{N}$.*
- (b) *If $f, g \in {}^\omega\mathbb{N}$ are both unbounded, then the games $G_f(\Omega_x, \Omega_x)$ and $G_g(\Omega_x, \Omega_x)$ are equivalent.*

Proof. For (a), we first note that a winning strategy for TWO in $G_f(\Omega_x, \Omega_x)$ is itself a winning strategy for TWO in $G_k(\Omega_x, \Omega_x)$, if TWO just ignores the finitely many innings $n \in \omega$ in which $f(n) > k$. For the converse, suppose that TWO has a winning strategy φ in $G_k(\Omega_x, \Omega_x)$. As the set $N = \{n \in \omega : f(n) = k\}$ is infinite, TWO can win an arbitrary play of $G_f(\Omega_x, \Omega_x)$ by skipping the innings $n \in \omega \setminus N$ and making use of φ in the innings $n \in N$ (considering, for the history of the play of $G_k(\Omega_x, \Omega_x)$ in which TWO applies φ , only the innings that are in N).

We now deal with ONE. Again, a winning strategy φ for ONE in the game $G_k(\Omega_x, \Omega_x)$ is a winning strategy for ONE in $G_f(\Omega_x, \Omega_x)$, for ONE can pretend that the only valid innings are the (cofinitely many) ones in $\{n \in \omega : f(n) \leq k\}$ and making use of φ in those innings. Conversely, suppose that ONE has a winning strategy φ in $G_f(\Omega_x, \Omega_x)$. Let $\{n_i : i \in \omega\}$ be an increasing enumeration of $\{n \in \omega : f(n) = k\}$. Now define a strategy for ONE in $G_k(\Omega_x, \Omega_x)$ as follows: in the inning $i \in \omega$, if the play so far is $(A_0, F_0, \dots, A_{i-1}, F_{i-1})$, ONE's move is $\varphi(G_0, G_1, \dots, G_{n_i-1})$, where $G_{n_j} = F_j$ for each $j < i$ and $G_m = \emptyset$ for all other values of m . Since φ is a winning strategy for ONE in $G_f(\Omega_x, \Omega_x)$, we have $\bigcup_{i \in \omega} F_i = \bigcup_{m \in \omega} G_m \notin \Omega_x$; therefore, this defines a winning strategy for ONE in $G_k(\Omega_x, \Omega_x)$.

For (b), we must show that

- $\text{TWO} \uparrow G_f(\Omega_x, \Omega_x)$ implies $\text{TWO} \uparrow G_g(\Omega_x, \Omega_x)$;
- $\text{ONE} \uparrow G_f(\Omega_x, \Omega_x)$ implies $\text{ONE} \uparrow G_g(\Omega_x, \Omega_x)$.

For the first implication, let φ be a winning strategy for TWO in $G_f(\Omega_x, \Omega_x)$, and let $(n_i)_{i \in \omega}$ be an increasing sequence in ω such that, for each $i \in \omega$, we have $g(n_i) \geq f(i)$. Then TWO can produce a winning strategy in $G_g(\Omega_x, \Omega_x)$ by playing along the innings in $\{n_i : i \in \omega\}$ only, making use of φ in those innings and ignoring the other innings.

Finally, for the second implication, let φ be a winning strategy for ONE in $G_f(\Omega_x, \Omega_x)$. Let $(m_i)_{i \in \omega}$ be an increasing sequence in ω such that, for each $i \in \omega$, we have $f(m_i) \geq g(i)$. We can now define a strategy ψ for ONE in $G_g(\Omega_x, \Omega_x)$ by setting $\psi(F_0, F_1, \dots, F_{i-1}) = \varphi(G_0, G_1, \dots, G_{m_i-1})$, where $G_{m_j} = F_j$ for each $j < i$ and $G_n = \emptyset$ for $n \notin \{m_j : j < i\}$. As in the last part of (a), it follows that $\bigcup_{i \in \omega} F_i = \bigcup_{n \in \omega} G_n \notin \Omega_x$, whence ψ is a winning strategy. \square

With Propositions 3.1 and 3.3 in mind, we henceforth adopt the following convention: whenever we write S_f and G_f , it should be understood that $f \in {}^\omega\mathbb{N}$ is unbounded.

Now we present all the variations in which we will be interested here. It is immediate that, for every space X , $p \in X$ and $k \in \mathbb{N}$, we have the following chain of implications:

$$\text{TWO} \uparrow G_k(\Omega_p, \Omega_p) \Rightarrow \text{TWO} \uparrow G_{k+1}(\Omega_p, \Omega_p) \Rightarrow \text{TWO} \uparrow G_f(\Omega_p, \Omega_p) \Rightarrow \text{TWO} \uparrow G_{\text{fin}}(\Omega_p, \Omega_p)$$

In what follows, we will show that none of these implications can be reversed in general.

Example 3.4. *There is a topological space showing that $\text{Two} \uparrow \mathbf{G}_{\text{fin}}(\Omega_x, \Omega_x)$ does not imply $\mathbf{S}_f(\Omega_x, \Omega_x)$.*

Proof. This is witnessed by $C_p(\mathbb{R})$. We have already remarked, right after Corollary 3.2, that $\text{Two} \uparrow \mathbf{G}_{\text{fin}}(\Omega_0, \Omega_0)$ on $C_p(\mathbb{R})$. On the other hand, $C_p(\mathbb{R})$ does not satisfy $\mathbf{S}_f(\Omega_0, \Omega_0)$ by Theorem 3.13 of [10]. \square

Another space satisfying the conditions in Example 3.4 can be found in Example 3.8 of [10].

Example 3.5. *There is a space showing that $\text{Two} \uparrow \mathbf{G}_f(\Omega_x, \Omega_x)$ does not imply $\mathbf{S}_1(\Omega_x, \Omega_x)$.*

Proof. This is witnessed by the following space, described in Example 3.7 of [10]. Consider, on $X = (\omega \times \omega) \dot{\cup} \{p\}$, the topology in which points of $\omega \times \omega$ are isolated and basic neighbourhoods of p are of the form $V_H = X \setminus \bigcup_{h \in H} \{(n, h(n)) : n \in \omega\}$, for H a finite subset of ${}^\omega\omega$. It is clear that X does not satisfy $\mathbf{S}_1(\Omega_p, \Omega_p)$, since each set $C_n = \{(n, m) : m \in \omega\}$ is in Ω_p . Note that a subset A of $\omega \times \omega$ satisfies $p \in \overline{A}$ if and only if $\sup\{|A \cap C_n| : n \in \omega\} = \aleph_0$; thus, we can obtain a winning Markov strategy for Two in the game $\mathbf{G}_f(\Omega_p, \Omega_p)$ on X as follows: in each inning $j \in \omega$, if $A_j \subseteq \omega \times \omega$ is the set played by ONE, let $n_j \in \omega$ be such that $|A_j \cap C_{n_j}| \geq f(j)$, and then declare Two's move to be $F_j \subseteq A_j \cap C_{n_j}$ with $|F_j| = f(j)$. \square

We have seen in Proposition 3.1 that the properties $\mathbf{S}_k(\Omega_x, \Omega_x)$ for $k \in \mathbb{N}$ are all equivalent. We also have seen in Example 2.10 that this is not the case between $\mathbf{G}_1(\Omega_x, \Omega_x)$ and $\mathbf{G}_2(\Omega_x, \Omega_x)$. Now we will see that all of the games $\mathbf{G}_k(\Omega_x, \Omega_x)$ for $k \in \mathbb{N}$ are distinct.

Example 3.6. *For each $k \in \mathbb{N}$, there is a countable space X_k with only one non-isolated point p on which $\text{One} \uparrow \mathbf{G}_k(\Omega_p, \Omega_p)$ and $\text{Two} \uparrow \mathbf{G}_{k+1}(\Omega_p, \Omega_p)$.*

Proof. This space is a variation of the space from Example 2.4. Write $\omega = \dot{\bigcup}_{s \in {}^{<\omega}\omega} N_s$ with each N_s infinite. For each $s \in {}^{<\omega}\omega$, fix a bijective enumeration $[N_s]^k = \{K_i^s : i \in \omega\}$. Now consider, on $X_k = \omega \dot{\cup} \{p\}$, the topology in which points in ω are isolated and basic neighbourhoods of p are of the form

$$X_k \setminus \bigcup_{g \in \mathcal{G}} \bigcup_{j \in \omega} K_{g(j)}^{g \upharpoonright j}$$

for $\mathcal{G} \subseteq {}^\omega\omega$ a finite set.

It is clear that ONE has a winning strategy in the game $\mathbf{G}_k(\Omega_p, \Omega_p)$ on X_k . We will now see that TWO has a winning strategy in the game $\mathbf{G}_{k+1}(\Omega_p, \Omega_p)$ on X_k .

Definition 3.7. *A fat branch is a subset of X_k of the form $\bigcup_{j \in \omega} K_{g(j)}^{g \upharpoonright j}$ for $g \in {}^\omega\omega$.*

The following is a simple but useful fact.

Lemma 3.8. *Let $Y \subseteq \omega$ be such that $p \in \overline{Y}$ in X_k . Then there is $B \subseteq Y$ with $|B| = k + 1$ that cannot be covered with a single fat branch.*

Proof. Suppose, by way of contradiction, that

(\dagger) each $B \subseteq Y$ with $|B| = k + 1$ is included in a fat branch.

This implies, in particular, that the set $S = \{s \in {}^{<\omega}\omega : Y \cap N_s \neq \emptyset\}$ is a chain in $<^\omega\omega$: if $s, t \in S$ were incompatible, we would contradict (\dagger) by picking $m \in Y \cap N_s$ and $n \in Y \cap N_t$ and then considering a set $B \subseteq Y$ with $|B| = k + 1$ satisfying $\{m, n\} \subseteq B$.

Case 1. S is infinite.

Let $g = \bigcup S \in {}^\omega\omega$. It follows from our hypothesis that $Y \not\subseteq \bigcup_{j \in \omega} K_{g(j)}^{g \upharpoonright j}$. Pick $n \in Y \setminus \bigcup_{j \in \omega} K_{g(j)}^{g \upharpoonright j}$, and let $s \in S$ be such that $n \in Y \cap N_s$. Now pick $t \in S$ with $s \subsetneq t$ and choose $m \in Y \cap N_t$. Then we need two distinct fat branches in order to cover the set $\{m, n\}$, since $n \notin K_{g(\text{dom}(s))}^{g \upharpoonright \text{dom}(s)} = K_{t(\text{dom}(s))}^s$. Again, this contradicts (\dagger) .

Case 2. S is finite.

Let $u = \max S$. By the same reasoning applied in Case 1, for each $s \in S$ with $s \subsetneq u$ we must have $Y \cap N_s \subseteq K_{u(\text{dom}(s))}^{u \upharpoonright \text{dom}(s)}$. It cannot be the case that $|Y \cap N_u| \geq k + 1$, since a set $B \subseteq Y \cap N_u$ with $|B| = k + 1$ would contradict (\dagger) . Thus, there is $i \in \omega$ such that $Y \cap N_u \subseteq K_i^u$. Now let $g \in {}^\omega\omega$ be such that $u \subseteq g$ and $g(\text{dom}(u)) = i$. Then $Y \subseteq \bigcup_{j \in \omega} K_{g(j)}^{g \upharpoonright j}$, which contradicts the hypothesis on Y . \square

Lemma 3.9. *TWO can play the game $G_{k+1}(\Omega_p, \Omega_p)$ on X_k in such a way that, after the inning $n \in \omega$, the set of all of the points picked by TWO includes a subset E_n with $|E_n| \geq n + k + 1$ such that no fat branch contains more than k points of E_n .*

Proof. For $Y \subseteq \omega$, define $Y^\downarrow = \bigcup_{n \in Y} \bigcup_{j \in \text{dom}(s_n)} K_{s_n(j)}^{s_n \upharpoonright j}$, where $s_n \in {}^{<\omega}\omega$ is such that $n \in N_{s_n}$.

We proceed by induction on n . For the initial inning, let TWO's move be the set E_0 given by Lemma 3.8. Now suppose that, after the inning $n \in \omega$, the set of all of the points picked by TWO includes a set E_n with $|E_n| \geq n + k + 1$ such that each fat branch contains at most k points of E_n , and let $A_{n+1} \in \Omega_p$.

Let $L = \{i \in \omega : \text{there is a fat branch containing both } i \text{ and some element of } E_n\}$. If $A_{n+1} \setminus L \in \Omega_p$, let TWO's move be $\{i\}$ for some $i \in A_{n+1} \setminus L$, and declare $E_{n+1} = E_n \cup \{i\}$. Let us then assume that $A_{n+1} \setminus L \notin \Omega_p$. It follows that $A_{n+1} \cap L \in \Omega_p$; furthermore, as E_n^\downarrow is finite, we may also assume that $A_{n+1} \cap E_n^\downarrow = \emptyset$.

For each $D \subseteq E_n$ with $|D| \leq k$, let $Z_D = \{i \in \omega : \text{there is a fat branch that includes } \{i\} \cup D\}$. Now let $d \leq k$ be maximal such that, for some $D \subseteq E_n$ with $|D| = d$, the set

$$A^* = \left(A_{n+1} \setminus \bigcup_{j=d+1}^k \bigcup_{C \in [E_n]^j} Z_C \right) \cap Z_D$$

is an element of Ω_p . Note that such a d must exist: if there is no $D \subseteq E_n$ with $|D| = k$ such that $A_{n+1} \cap Z_D \in \Omega_p$, then $A_{n+1} \setminus \bigcup_{C \in [E_n]^k} Z_C$ must be in Ω_p ; now, if there is no $D \subseteq E_n$ with $|D| = k - 1$ such that $(A_{n+1} \setminus \bigcup_{C \in [E_n]^k} Z_C) \cap Z_D \in \Omega_p$, then $(A_{n+1} \setminus \bigcup_{C \in [E_n]^k} Z_C) \setminus \bigcup_{C \in [E_n]^{k-1}} Z_C$ must be in Ω_p ; and so forth. This process must stop at some $d > 0$ since $A_{n+1} \cap L \in \Omega_p$.

Let B be the set obtained by applying Lemma 3.8 to the set A^* . We will show that, by letting TWO's move in this inning be B , the set $E_{n+1} = (E_n \setminus D) \cup B$ will satisfy the condition required by the induction.

Let $G \subseteq \omega$ be a fat branch. If $G \cap (E_n \setminus D) = \emptyset$, then $G \cap E_{n+1} = G \cap B$ has no more than k points. If $G \cap B = \emptyset$, then the set $G \cap E_{n+1}$ is a subset of $G \cap E_n$,

hence it has no more than k points. Let us now deal with the case in which the sets $G \cap (E_n \setminus D)$ and $G \cap B$ are both non-empty.

Let $i \in G \cap (E_n \setminus D)$ and $m \in G \cap B$ be arbitrary. Since $A_{n+1} \cap E_n^\perp = \emptyset$, it follows that $B \cap \{i\}^\perp = \emptyset$; thus, as $m \in G \cap B$ and $i \in G$, it must be the case that $i \in N_t$ for some $t \subseteq s$, where $s \in {}^{<\omega}\omega$ is such that $m \in N_s$. Let $g \in {}^\omega\omega$ be such that $G = \bigcup_{j \in \omega} K_{g(j)}^{g \upharpoonright j}$, and let $h \in {}^\omega\omega$ be such that the fat branch $H = \bigcup_{j \in \omega} K_{h(j)}^{h \upharpoonright j}$ includes the set $\{m\} \cup D$. As $m \in G \cap H$, it follows that $g \cap h \supseteq s$. Let $j_0 = \text{dom}(t)$.

We claim that $t = s$. Suppose, to the contrary, that $t \subsetneq s$. Then $i \in K_{g(j_0)}^t = K_{h(j_0)}^t \subseteq H$, which implies that H is a fat branch that includes $C = \{i\} \cup D$. But then $m \in Z_C$, since m is also an element of H . This contradicts the fact that $m \in B \subseteq A^*$.

Thus, as i and m are arbitrary, we have proved that $(G \cap (E_n \setminus D)) \cup (G \cap B) \subseteq N_t$. This implies that $G \cap E_{n+1} \subseteq G \cap N_t = K_{g(j_0)}^t$, which in turn yields $|G \cap E_{n+1}| \leq k$, as required. \square

The existence of a winning strategy for TWO in the game $G_{k+1}(\Omega_p, \Omega_p)$ on X_k is an immediate consequence of Lemma 3.9. \square

We will now see a space in which all of the games considered in this work are undetermined.

Example 3.10. *There is a countable space with only one non-isolated point p on which ONE $\nrightarrow G_1(\Omega_p, \Omega_p)$ and TWO $\nrightarrow G_{\text{fin}}(\Omega_p, \Omega_p)$.*

Proof. This is essentially the space from Example 2.11 of [13].

Consider, on the set $T = {}^{<\omega}2$, the topology generated by the base $\{\{s\} : s \in {}^{<\omega}2\} \cup \{V_s : s \in {}^{<\omega}2\}$, where $V_s = \{t \in T : s \subseteq t\}$ for each $s \in {}^{<\omega}2$. The subspace ${}^\omega 2$ of T is homeomorphic to the Cantor set; let then $B \subseteq {}^\omega 2$ be a Bernstein set in ${}^\omega 2$ (see e.g. [15, Theorem 11.4]).

We will now consider the space $X_B = ({}^{<\omega}2) \dot{\cup} \{p\}$ in which every $s \in {}^{<\omega}2$ is isolated and basic neighbourhoods of p are of the form $U_F = X_B \setminus \bigcup_{f \in F} \{f \upharpoonright j : j \in \omega\}$ for $F \in [B]^{<\aleph_0}$.

Remark 3.11. By essentially the same argument from Theorem 1 of [9], TWO (resp. ONE) has a winning strategy in the game $G_1(\Omega_x, \Omega_x)$ played on X if and only if ONE (resp. TWO) has a winning strategy in the game $G_{O,P}^c(X, x)$, defined as follows. In each inning $n \in \omega$, ONE chooses an open neighbourhood V_n of x in X , and then TWO picks a point $x_n \in V_n$. The winner is ONE if $\{x_n : n \in \omega\} \in \Omega_x$, and TWO otherwise.

Proposition 3.12 (Gruenhage [13]). TWO $\nrightarrow G_{O,P}^c(X_B, p)$.

We should point out that, although the space X_B in our construction is not quite the same as the one exhibited in Example 2.11 of [13], adapting the proof in Gruenhage's paper to the space X_B is quite straightforward.

By Remark 3.11, Theorem 3.12 is equivalent to the statement that ONE $\nrightarrow G_1(\Omega_p, \Omega_p)$ on X_B . Thus, in order to conclude that all of the games we are considering are undetermined on X_B at p , we must show that TWO $\nrightarrow G_{\text{fin}}(\Omega_p, \Omega_p)$ on X_B . In order to prove this, we will need a few auxiliary results.

Lemma 3.13. *Let $A \subseteq {}^{<\omega}2$. The following statements are equivalent:*

- (a) $p \in \overline{A}$ in X_B ;

- (b) A includes an infinite antichain or there is $g \in {}^\omega 2 \setminus B$ such that the set $A \cap \{g \restriction j : j \in \omega\}$ is infinite.

Proof. The implication (b) \Rightarrow (a) is clear. For the converse, suppose that $p \in \overline{A}$ in X_B and that every antichain included in A is finite. Then the same holds for the set $A^- = \bigcup_{s \in A} \{s \restriction j : j \leq \text{dom}(s)\} \supseteq A$. Since A^- is an infinite subtree of ${}^{<\omega} 2$, it must have at least one infinite branch.

Claim. The set $C = \{f \in {}^\omega 2 : \forall j \in \omega (f \restriction j \in A^-)\}$ is finite.

For each $f \in C$, there must be some $j_f \in \omega$ such that $\{s \in A^- : s \supseteq f \restriction j_f\} = \{f \restriction j' : j' \leq j_f\}$, for otherwise A^- would include an infinite antichain. This implies that $\{f \restriction j_f : f \in C\} \subseteq A^-$ is an antichain, whence C must be finite.

Thus, as $U_{C \cap B}$ is an open neighbourhood of p in X_B , it follows that the set $D = A^- \cap U_{C \cap B}$ is such that $p \in \overline{D}$ in X_B . Now let $E = \bigcup_{s \in D} \{s \restriction j : j \leq \text{dom}(s)\} \subseteq A^-$. Also the set E is an infinite subtree of ${}^{<\omega} 2$, hence it must have an infinite branch – say, $\{g \restriction j : j \in \omega\}$ for $g \in {}^\omega 2$. The procedure for obtaining E from A^- guarantees that $g \notin B$. As $\{g \restriction j : j \in \omega\} \subseteq A^-$, it follows from the definition of A^- that $\{j \in \omega : g \restriction j \in A\}$ is infinite. \square

Lemma 3.14. *Let $R \subseteq B$ be such that, for every $g \in {}^\omega 2 \setminus B$, there is $j \in \omega$ such that no $f \in R$ extends $g \restriction j$. Then R is nowhere dense in ${}^\omega 2$.*

Proof. Fix, for each $g \in {}^\omega 2 \setminus B$, a $j_g \in \omega$ such that $\forall f \in R (g \restriction j_g \not\subseteq f)$. Then

$$\bigcup_{g \in {}^\omega 2 \setminus B} V_{g \restriction j_g} \cap {}^\omega 2$$

is an open subset of ${}^\omega 2$ which, furthermore, is dense in ${}^\omega 2$ – since ${}^\omega 2 \setminus B$ is dense in ${}^\omega 2$. It follows from the choice of the j_g s that R is disjoint from this dense open set. Thus, R is nowhere dense in ${}^\omega 2$. \square

Definition 3.15. *Let X be a topological space and $x \in X$. A family \mathcal{C} of nonempty open subsets of X is simple at x if every $A \subseteq X$ with $x \in \overline{A}$ includes a finite set that intersects every element of \mathcal{C} .*

The following result parallels Theorems 2.10 and 2.11 of [6], and is inspired by Theorem 1 of [23].

Proposition 3.16. *Let X be a topological space and $x \in X$. Consider the following statements:*

- (a) every local base for X at x is a countable union of simple families;
- (b) τ_x is a countable union of simple families;
- (c) there is a local base for X at x that is a countable union of simple families;
- (d) $\text{Two} \uparrow \text{G}_{\text{fin}}(\Omega_x, \Omega_x)$ on X .

Then (a) \Leftrightarrow (b) \Leftrightarrow (c) \Rightarrow (d). Moreover, if X is countable, then the four statements are equivalent.

Proof. We prove the second part only – namely, the implication (d) \Rightarrow (b). Let then X be a countable space on which there is a winning strategy φ for TWO in the game $\text{G}_{\text{fin}}(\Omega_x, \Omega_x)$.

Define

$$\mathcal{U}_\emptyset = \bigcap_{A \in \Omega_x} \{U \in \tau_x : U \cap \varphi(A) \neq \emptyset\}.$$

As X is countable, the set $\{\varphi(A) : A \in \Omega_x\} \subseteq [X]^{<\aleph_0}$ is countable; let $\mathcal{A}_\emptyset \subseteq \Omega_x$ be such that $\{\varphi(A) : A \in \Omega_x\} = \{\varphi(A_i^\emptyset) : i \in \omega\}$. Now, for each $(i_0) \in {}^1\omega$, let

$$\mathcal{U}_{(i_0)} = \bigcap_{A \in \Omega_x} \{U \in \tau_x : U \cap \varphi(A_{i_0}^\emptyset, A) \neq \emptyset\}.$$

Let $\{A_i^{(i_0)} : i \in \omega\} \subseteq \Omega_x$ be such that $\{\varphi(A_{i_0}^\emptyset, A) : A \in \Omega_x\} = \{\varphi(A_{i_0}^\emptyset, A_i^{(i_0)}) : i \in \omega\}$. For each $(i_0, i_1) \in {}^2\omega$, define

$$\mathcal{U}_{(i_0, i_1)} = \bigcap_{A \in \Omega_x} \{U \in \tau_x : U \cap \varphi(A_{i_0}^\emptyset, A_{i_1}^{(i_0)}, A) \neq \emptyset\},$$

and pick $\{A_i^{(i_0, i_1)} : i \in \omega\} \subseteq \Omega_x$ satisfying $\{\varphi(A_{i_0}^\emptyset, A_{i_1}^{(i_0)}, A) : A \in \Omega_x\} = \{\varphi(A_{i_0}^\emptyset, A_{i_1}^{(i_0)}, A_i^{(i_0, i_1)}) : i \in \omega\}$. Then define

$$\mathcal{U}_{(i_0, i_1, i_2)} = \bigcap_{A \in \Omega_x} \{U \in \tau_x : U \cap \varphi(A_{i_0}^\emptyset, A_{i_1}^{(i_0)}, A_{i_2}^{(i_0, i_1)}, A) \neq \emptyset\}$$

for each $(i_0, i_1, i_2) \in {}^3\omega$, and so on. By proceeding in this fashion, we construct \mathcal{U}_s for every $s \in {}^{<\omega}\omega$, by induction on $\text{dom}(s)$.

Note that each family \mathcal{U}_s is simple. The proof will be finished once we show that $\bigcup_{s \in {}^{<\omega}\omega} \mathcal{U}_s = \tau_x$.

Suppose, by way of contradiction, that there is $V \in \tau_x \setminus \bigcup_{s \in {}^{<\omega}\omega} \mathcal{U}_s$. We can then recursively pick i_0, i_1, i_2, \dots in ω such that

$$V \cap \varphi(A_{i_0}^\emptyset, A_{i_1}^{(i_0)}, \dots, A_{i_n}^{(i_0, \dots, i_{n-1})}) = \emptyset$$

for each $n \in \omega$. But then

$$(\varphi(A_{i_0}^\emptyset), \varphi(A_{i_0}^\emptyset, A_{i_1}^{(i_0)}), \dots, \varphi(A_{i_0}^\emptyset, A_{i_1}^{(i_0)}, \dots, A_{i_n}^{(i_0, \dots, i_{n-1})}), \dots)$$

is the sequence of Two's moves in a play of $\mathbf{G}_{\text{fin}}(\Omega_x, \Omega_x)$ in which Two makes use of the strategy φ and loses. This contradicts the fact that φ is a winning strategy. \square

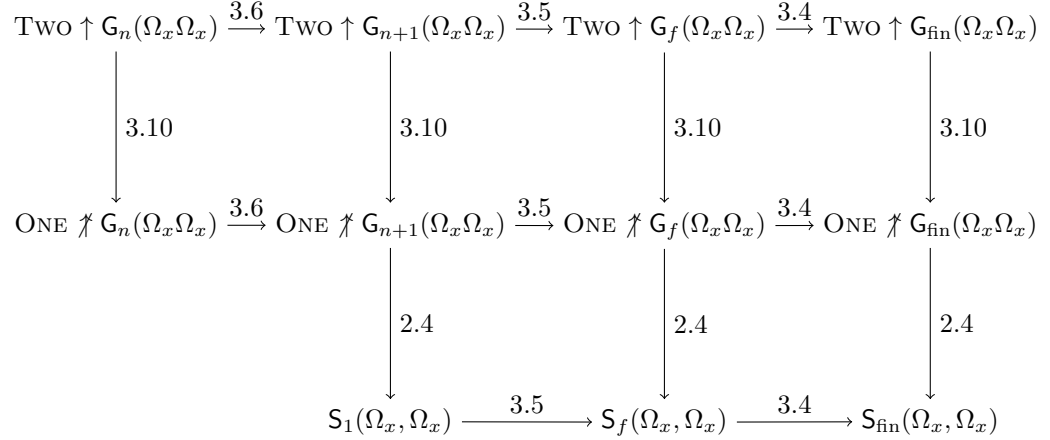
We can now proceed to the proof that Two $\nVdash \mathbf{G}_{\text{fin}}(\Omega_p, \Omega_p)$ on X_B .

We will make use of Proposition 3.16. Suppose, towards a contradiction, that $\{U_H : H \in [B]^{<\aleph_0}\} = \bigcup_{n \in \omega} \mathcal{F}_n$, where \mathcal{F}_n is simple for each $n \in \omega$.

Let $n \in \omega$ be arbitrary. Define $\mathcal{H}_n = \{H \in [B]^{<\aleph_0} : U_H \in \mathcal{F}_n\}$. As \mathcal{F}_n is simple, it follows from Lemma 3.13 that every $A \subseteq {}^{<\omega}2$ satisfying $|A \cap \{g \restriction j : j \in \omega\}| \geq \aleph_0$ for some $g \in ({}^\omega 2) \setminus B$ includes a finite subset A_0 that meets every element of \mathcal{F}_n . By considering the particular case $A = \{g \restriction j : j \in \omega\}$ of this statement, we obtain that, for every $g \in ({}^\omega 2) \setminus B$, there is a finite $J_g \subseteq \omega$ such that, for every $H \in \mathcal{H}_n$, there is some $j' \in J_g$ satisfying $\forall h \in H (g \restriction j' \not\subseteq h)$. Therefore, for every $g \in ({}^\omega 2) \setminus B$, it follows that $j_g = \max J_g$ satisfies $g \restriction j_g \not\subseteq h$ for all $h \in \bigcup \mathcal{H}_n$. Hence, $\bigcup \mathcal{H}_n$ is nowhere dense in ${}^\omega 2$ by Lemma 3.14.

Now, as $B \subseteq {}^\omega 2$ is not meagre (see e.g. [20, pp. 29–30]), there is $x \in B \setminus \bigcup_{n \in \omega} \bigcup \mathcal{H}_n$. The set $U_{\{x\}}$ must be an element of \mathcal{F}_n for some $n \in \omega$ – which yields $\{x\} \in \mathcal{H}_n$, thus contradicting the choice of x . \square

The results obtained so far can be summarized by the following diagram:



In the diagram, the arrows marked with 2.4 indicate that Example 2.4 shows that the converse of the corresponding implication does not hold; and so forth.

4. NEW DIRECTIONS AND OPEN PROBLEMS

The selective topological properties we considered in this work are the “countable tightness” particular cases of a broader (non-topological) framework introduced by M. Scheepers in [24], which gave rise to a field of research that today is known as the study of *selection principles*.

Definition 4.1 (Scheepers [24]). *Let \mathcal{A}, \mathcal{B} be nonempty families of nonempty sets. $S_1(\mathcal{A}, \mathcal{B})$ and $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$ denote, respectively, the following statements:*

- $S_1(\mathcal{A}, \mathcal{B}) \equiv$ *for every sequence $(A_n)_{n \in \omega}$ of elements of \mathcal{A} , there is a sequence $(b_n)_{n \in \omega}$ such that $b_n \in A_n$ for each $n \in \omega$ and $\{b_n : n \in \omega\} \in \mathcal{B}$;*
 $S_{\text{fin}}(\mathcal{A}, \mathcal{B}) \equiv$ *for every sequence $(A_n)_{n \in \omega}$ of elements of \mathcal{A} , there is a sequence $(F_n)_{n \in \omega}$ of finite sets such that $F_n \subseteq A_n$ for each $n \in \omega$ and $\bigcup_{n \in \omega} F_n \in \mathcal{B}$.*

Although in this work we have concentrated on the instance $(\mathcal{A}, \mathcal{B}) = (\Omega_x, \Omega_x)$ of the property schemas above, there are various contexts in which properties of this kind arise naturally. A typical example is provided by the classical Rothberger [21] and Menger [14] covering properties, which can be expressed in terms of selection principles as $S_1(\mathcal{O}, \mathcal{O})$ and $S_{\text{fin}}(\mathcal{O}, \mathcal{O})$ respectively; here, \mathcal{O} stands for the family of all of the open covers of a given topological space.

Some other selection principles have been added to S_1 and S_{fin} in the literature, such as the selection principle S_f from Section 3 – which has appeared in e.g. the following result, proved in Lemma 3.12 of [10] and Lemma 3 of [8] (see also [30, Appendix A]):

Proposition 4.2 (García-Ferreira–Tamariz-Mascarúa [10], Bukovský–Ciesielski [8]). *Let $f \in {}^\omega \mathbb{N}$ be arbitrary. Then the properties $S_1(\mathcal{O}, \mathcal{O})$ and $S_f(\mathcal{O}, \mathcal{O})$ are equivalent.*

Each selection principle has game naturally associated to it. For example, in the original definition due to Scheepers, we have:

Definition 4.3 (Scheepers [25]). *Let \mathcal{A}, \mathcal{B} be nonempty families of nonempty sets. $G_1(\mathcal{A}, \mathcal{B})$ and $G_{\text{fin}}(\mathcal{A}, \mathcal{B})$ denote, respectively, the following games.*

- In each inning $n \in \omega$ of $G_1(\mathcal{A}, \mathcal{B})$, ONE chooses $A_n \in \mathcal{A}$, and then TWO picks $b_n \in A_n$. The winner is TWO if $\{b_n : n \in \omega\} \in \mathcal{B}$, and ONE otherwise.
- In each inning $n \in \omega$ of $G_{\text{fin}}(\mathcal{A}, \mathcal{B})$, ONE chooses $A_n \in \mathcal{A}$, and then TWO picks a finite subset $F_n \subseteq A_n$. The winner is TWO if $\bigcup_{n \in \omega} F_n \in \mathcal{B}$, and ONE otherwise.

Though the selection principles S_f and S_k have already occurred in the literature, to our knowledge their game versions G_f and G_k have not been object of study thus far. The fact that such counterintuitive differences between games such as G_k and G_{k+1} spring in a typically infinitary context such as topological spaces – in which many general properties are not sensitive to changes from a finite number to another – suggests that investigating other instances of these games might lead to a whole new class of interesting results.

Problem 4.4. *What can be said about the relation between the various games $G_k(\mathcal{A}, \mathcal{B})$, $G_f(\mathcal{A}, \mathcal{B})$ and $G_{\text{fin}}(\mathcal{A}, \mathcal{B})$ – and their associated selective properties – for other pairs $(\mathcal{A}, \mathcal{B})$?*

As we have seen in this work, the games of form $G_k(\Omega_x, \Omega_x)$ bear differences between them that are not verified in their selective counterparts $S_k(\Omega_x, \Omega_x)$. Since, by Proposition 4.2, the class of mutually equivalent selective properties of form $S_{\square}(\mathcal{O}, \mathcal{O})$ is even greater, one may wonder whether the differences between the games are also verified in this case.

Problem 4.5. *For $k \in \mathbb{N}$, is there a space on which the games $G_k(\mathcal{O}, \mathcal{O})$ and $G_{k+1}(\mathcal{O}, \mathcal{O})$ are not equivalent? What about their relation to the game $G_f(\mathcal{O}, \mathcal{O})$ for $f \in {}^\omega \mathbb{N}$?*

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